

# The effect of angular momentum conservation in the phase transitions of collapsing systems.

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## Abstract

The effect of angular momentum conservation in microcanonical thermodynamics is considered. This is relevant in gravitating systems, where angular momentum is conserved and the collapsing nature of the forces makes the microcanonical ensemble the proper statistical description of the physical processes. The microcanonical distribution function with non-vanishing angular momentum is obtained as a function of the coordinates of the particles. As an example, a simple model of gravitating particles, introduced by Thirring long ago, is worked out. The phase diagram contains three phases: for low values of the angular momentum  $L$  the system behaves as the original model, showing a complete collapse at low energies and an entropy with a convex intruder. For intermediate values of  $L$  the collapse at low energies is not complete and the entropy still has a convex intruder. For large  $L$  there is neither collapse nor anomalies in the thermodynamical quantities. A short discussion of the extension of these results to more realistic situations is exposed.

## 1 Introduction.

Conventional thermodynamics applies to systems whose forces saturate and therefore macroscopic parts have negligible interactions. If this is the case, the macroscopically conserved quantities are extensive, the thermodynamic potentials are homogeneous functions of degree one of these quantities and large systems can be studied in the thermodynamical limit. These properties hold if the so called stability condition is verified. For a system of  $N$  classical particles interacting through a two body potential,  $\phi(r)$ , the stability condition states that there must exist a positive constant  $E_0$  such that, for any configuration  $\{\mathbf{r}_1, \dots, \mathbf{r}_N\}$ , the following inequality is obeyed [1]:

$$\Phi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{2} \sum_{i \neq j} \phi(|\mathbf{r}_i - \mathbf{r}_j|) \geq -NE_0. \quad (1)$$

These cases constitute very specific situations. For example, (1) holds for potentials repulsive enough at short distances, like the Lennard-Jones potential, but it is clearly violated for gravitating systems. When the stability condition does not hold, the system undergoes a phase transition from a high energy gas phase to a collapsing phase at low energies. This low energy regime is not a proper thermodynamical phase, since it is not homogeneous, but, nevertheless, we shall use the name collapsing phase throughout this paper.

Microcanonical thermodynamics has been recognized as the statistical description of systems which suffer fragmentation and clustering [2], since there seems to be problems to describe spatial inhomogeneities within the canonical formalism. These phenomena, fragmentation and clustering, appear in many different branches of physics, from nuclear physics [3] and atomic clusters [4] to astrophysics [5]. The task of microcanonical thermodynamics is to compute the entropy of a given system as a function of the macroscopically conserved quantities. It is intuitively clear that the entropy of a system which undergoes a phase transition associated to spatial inhomogeneities like clustering or collapsing will depend crucially on the value of its total angular momentum, if conserved, since the spatial distribution will do. (See ref. [6] for a similar discussion in a different context). Angular momentum is of particular importance in astrophysics: the formation of a single star or of a binary system, the birth of a solar system around a star or the merger of galaxies in clusters is, to a large extent, determined by the value of the angular momentum.

It is well known that, as a consequence of the virial theorem, gravitating systems have negative specific heat [7, 8]. In fact, it seems that the common

feature of all these collapsing systems is the appearance of an interval of energies where the microcanonical specific heat is negative [9, 10]. Since the canonical specific heat is always positive, the microcanonical ensemble gives the proper statistical description of these systems.

A system of classical gravitating particles interacting via the newtonian potential has an infinite entropy, due to both short and long distance singularities [7]. It is clear that the short distance singularity of this potential is not physical, since new physics (like quantum mechanics) appears at small scales. Hence, there should exist a natural short distance cut-off. The long distance singularity has a different nature. In principle, one would say that gravity should be able to bind the system. A little caveat, however, easily convinces oneself that, at least from the statistical point of view, the system prefers to evaporate than to remain bound. One needs a box (the long distance cut-off) to keep the system confined, but no such a box appears in nature. We have to consider it as an artifact making the statistical description of the system possible, which will be sensible if the evaporation rate is small. The box breaks translational invariance, and therefore the momentum is not conserved. Since we are interested in keeping angular momentum conserved, we must deal with spherical boxes, in order to maintain the rotational symmetry exact.

The paper is organized as follows: in section 2 we compute the microcanonical distribution with conserved angular momentum, by integrating out the momenta in the phase space. The formula obtained gives a microcanonical weight suitable for Monte Carlo simulations, and allows also for analytical approaches relying in mean field methods. In sec. 3 we briefly derive the mean field equations for systems which conserve angular momentum. Sec. 4 is devoted to the discussion of a simple model, introduced by Thirring [9], which mimics the main features of gravitating systems with great success. The model is extended to take into account the conservation of angular momentum. The article ends with a summary of the conclusions, in sec 5.

## 2 The microcanonical distribution.

Consider a system of  $N$  classical particles whose interactions are given by a general potential energy,  $\Phi$ , depending only on the position of the particles. The hamiltonian reads:

$$\mathcal{H} = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m_i} + \Phi(\mathbf{r}_1, \dots, \mathbf{r}_N) . \quad (2)$$

If the system is isolated and the potential energy is translationally and rotationally invariant, the energy, momentum and angular momentum will be conserved, and, as a consistency condition, the center of mass will move with constant velocity. Without any loss of generality, we can take the total momentum zero and then the center of mass is fixed, for instance, at the origin of coordinates. Assuming some kind of ergodicity, the microcanonical distribution will give the statistical description of the system. The entropy  $S$  is given by the Boltzmann formula:

$$S(E, L, N) = \log W(E, L, N), \quad (3)$$

where  $E$  is the total energy and  $L$  the modulo of the angular momentum, since, by symmetry, the entropy must be independent of the direction of the vector  $\mathbf{L}$ ;  $W$  is the volume of the phase space shell defined by the orbits with given  $E$  and  $\mathbf{L}$ . Notice that we took the Boltzmann constant equal to unity, and the entropy dimensionless. Hence, the temperature is measured in units of energy.

To avoid too cumbersome expressions, let us consider the case in which only the angular momentum is conserved. If also the linear momentum is conserved, the derivation of the microcanonical distribution is similar and the result will be reported at the end of the section. The volume of the relevant cell of the phase space can be computed starting from its definition:

$$W(E, L, N) = \frac{1}{N!} \int \left( \prod_{i=1}^N \frac{d^3 r_i d^3 p_i}{2\pi\hbar} \right) \delta(E - \mathcal{H}) \delta^{(3)}(\mathbf{L} - \sum_i \mathbf{r}_i \wedge \mathbf{p}_i). \quad (4)$$

Integrating out the  $\mathbf{p}$  variables, we will obtain a non-singular microcanonical probability distribution depending only on the spatial configuration  $\{\mathbf{r}_1, \dots, \mathbf{r}_N\}$ . In this way, we shall get an expression suitable for microcanonical Monte Carlo simulations.

To perform easily the  $\mathbf{p}$  integration in (4), let us define

$$\Pi(\bar{E}, L, N, \{\mathbf{r}\}) = \int \prod_{i=1}^N d^3 p_i \delta(\bar{E} - \sum_i \frac{p_i^2}{2m_i}) \delta^{(3)}(\mathbf{L} - \sum_i \mathbf{r}_i \wedge \mathbf{p}_i), \quad (5)$$

where we used the notation  $\bar{E} = E - \Phi(\mathbf{r}_1, \dots, \mathbf{r}_N)$ . It is convenient to take the Laplace transform of  $\Pi$  respect to  $\bar{E}$ ,

$$\tilde{\Pi}(s, L, N, \{\mathbf{r}\}) = \int_0^\infty d\bar{E} e^{-s\bar{E}} \Pi(\bar{E}, L, N, \{\mathbf{r}\}), \quad (6)$$

with  $\text{Re } s > 0$ . Introducing the following representation for the remaining Dirac delta:

$$\delta(x) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{i\omega x}, \quad (7)$$

we obtain

$$\begin{aligned} \tilde{\Pi}(s, L, N, \{\mathbf{r}\}) &= \int \frac{d^3\omega}{(2\pi)^3} \exp(i\omega \cdot \mathbf{L}) \int \prod_{i=1}^N d^3p_i \\ &\quad \exp \left( -s \sum_i \frac{p_i^2}{2m_i} - i \sum_i \boldsymbol{\omega} \cdot (\mathbf{r}_i \wedge \mathbf{p}_i) \right). \end{aligned} \quad (8)$$

The integral in  $\mathbf{p}$  is now gaussian and can be readily performed. What shall remain is again a gaussian integral in  $\boldsymbol{\omega}$ ; there is no difficulty in evaluating it. After some algebra it is found:

$$\tilde{\Pi}(s, L, N, \{\mathbf{r}\}) = C (\det I)^{-1/2} \frac{e^{-s \frac{1}{2} L^T I^{-1} L}}{s^{(3N-3)/2}}, \quad (9)$$

where  $L^T I^{-1} L = \sum_{\alpha\beta=1}^3 L_\alpha I_{\alpha\beta}^{-1} L_\beta$ , the matrix  $I$  is the inertial tensor respect to the origin,

$$I_{\alpha\beta} = \sum_{i=1}^N m_i \left( r_i^2 \delta_{\alpha\beta} - r_i^\alpha r_i^\beta \right), \quad (10)$$

with  $\alpha, \beta = 1, 2, 3$  labeling the coordinates, and  $C = (2\pi)^{\frac{3N-3}{2}} \prod_i m_i^{3/2}$  is a constant.

The inverse Laplace transform of (9) can be found in any book of integral transform tables [11]. It is:

$$\frac{e^{-sb}}{s^\nu} \sim \begin{cases} 0 & \text{if } 0 < \bar{E} < b \\ \frac{1}{\Gamma(\nu)} (\bar{E} - b)^{\nu-1} & \text{if } \bar{E} > b \end{cases} . \quad (11)$$

Therefore

$$\Pi(\bar{E}, L, N, \{\mathbf{r}\}) = \frac{C}{\Gamma(\frac{3N-3}{2})} (\det I)^{-1/2} \left( \bar{E} - \frac{1}{2} L^T I^{-1} L \right)^{\frac{3N-5}{2}} \quad (12)$$

if  $\bar{E} > \frac{1}{2} L^T I^{-1} L$ , and it vanishes otherwise.

After the  $\mathbf{p}$  integration the volume of the phase space is given by:

$$W(E, L, N) = \tilde{C} \int \left( \prod_{i=1}^N \frac{d^3 r_i}{2\pi\hbar} \right) \frac{1}{\sqrt{\det I}} \left( E - \frac{1}{2} L^T I^{-1} L - \Phi \right)^{\frac{3N-5}{2}}, \quad (13)$$

where  $\tilde{C} = C/[N! \Gamma((3N-3)/2)]$ . The thermodynamical quantities can be computed numerically from the last integral by, for instance, using a suitable Monte Carlo algorithm. Eq. (13) can be also used to derive microcanonical equations for mean field approximations.

If the linear momentum is also conserved, the same expressions hold, changing the exponent in eq. (13) by  $(3N-8)/2$ , the constant  $C$  by

$$C = (2\pi)^{\frac{3N-6}{2}} (\sum_i m_i)^{-3/2} \prod_i m_i^{3/2} \quad (14)$$

and the argument of the  $\Gamma$  function appearing in  $\tilde{C}$  by  $(3N-6)/2$ . In this case, the inertial tensor  $I$  refers to the center of mass, which can be taken at the origin, and then eq. (10) holds. Notice that, if the number of particles is large, the momentum conservation gives negligible differences.

If the particles have some internal spin, then the intrinsic inertial moment of each particle must be added to the orbital inertial moment, and the exponent in (13) must be properly modified to take into account the number of intrinsic degrees of freedom. The derivation of a microcanonical weight for these more

general cases is identical to that outlined in this section and presents no additional difficulty. As a last remark, we stress that the results of this section, i.e. eq. (13), apply to stable as well as to unstable systems.

### 3 Mean Field theory.

If the number of particles  $N$  is very large, one can expect that the force that a particle undergoes will be more sensitive to the mean particle distribution of the system than to the fluctuations around it. Then one can think that the thermodynamics of the system depends only on the mean particle density. In this section, we shall give a derivation of the mean field equations using this hypothesis (cf. ref. [12]). For reasons which will become clear in the following, the results of this section apply only to unstable systems.

Let us start with the expression given by eq. (13). To perform the integral in  $\mathbf{r}$ , we divide the integration region in  $\Lambda$  cells of volume  $a^3$ . The integral over  $d^3r_i$  becomes a sum over all possible cells. We have then  $N$  of such sums. For each configuration  $\{\mathbf{r}\}$  we can define a function  $n(i)$ , with  $i = 1, \dots, \Lambda$ , which counts the number of particles in the  $i$ -th cell. It is not difficult to see that the initial sum over the position of the particles can be reorganized as a sum over the number of particles in each cell, as follows:

$$W(E, L, N) = A \sum_{n(1)=0}^N \dots \sum_{n(\Lambda)=0}^N \delta \left( \sum_{i=1}^{\Lambda} n(i) - N \right) \frac{N!}{\prod_{i=1}^{\Lambda} n(i)!} \times \frac{1}{\sqrt{\det I}} \left( E - \frac{1}{2} L^T I^{-1} L - \Phi \right)^{\frac{3N-5}{2}}. \quad (15)$$

where  $A$  is a constant which contains  $\tilde{C}$ , the elementary phase space volume  $2\pi\hbar$  and the volume of the cell,  $a^3$ . The combinatorial factor arises since a certain amount (= this factor) of configurations of particle positions give the same cell occupation distribution  $\{n(i)\}$ . Introducing the particle density  $\rho(\mathbf{r}_i) = n(i)/(Na^3)$ , where  $\mathbf{r}_i$  denotes the position of the center of the  $i$ -th cell, we can write (15) as the following functional integral:

$$W(E, L, N) = \int [d\rho] \delta \left( \int d^3r \rho(\mathbf{r}) - 1 \right) \times$$

$$\exp \left\{ N \left[ - \int d^3r \rho(\mathbf{r}) [\log \rho(\mathbf{r}) - 1] + \frac{3}{2} \log \left( E - \frac{1}{2} L^T I^{-1} L - \Phi \right) \right] \right\} \quad (16)$$

where we have substituted the factorials by its asymptotic form since we consider  $N$  large, and ignored some irrelevant constants. We have also neglected the term  $\sqrt{\det I}$ , which is of order  $1/N$ . Now, we have to clarify how the argument of the logarithm scales with  $N$ . For a given discretization, it is clear that the potential energy  $\Phi$  scales as  $N^2$ , irrespective of whether the potential is stable or unstable. With  $N$  fixed, in the continuum limit,  $a \rightarrow 0$ , the potential energy scales with  $N$  if the potential is stable, and a proper thermodynamical limit can be taken. For unstable potentials, however, the potential energy scales with  $N^2$  still in the continuum limit. The usual thermodynamical limit does not exist for this kind of systems. However, we can take  $N \rightarrow \infty$  and send the masses and the coupling constants in the potential energy to zero, keeping  $Nm = M$  and  $N^2\phi(\mathbf{r}, \mathbf{r}')$  constant. In this case, the energy  $E$  is not scaled; the thermodynamical functions depend on the total energy, instead of on the energy per particle. However, as can be seen from (16), the entropy scales with  $N$ . This is the scaling to be considered in this paper, and applies naturally to unstable systems.

The functions  $I$  and  $\Phi$  in (16) depend on  $\rho(\mathbf{r}_i)$  through a suitable discretization. For very large  $N$  and very small  $a$ , they can be expressed as:

$$\begin{aligned} I_{\alpha\beta} [\rho] &= M \int d^3r \rho(\mathbf{r}) (r^2 \delta_{\alpha\beta} - r_\alpha r_\beta) \\ \Phi [\rho] &= \int d^3r d^3r' \phi(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}) \rho(\mathbf{r}') . \end{aligned} \quad (17)$$

The functional integral (16) is defined by the previous discretization. There are two limits to be considered in (16),  $a \rightarrow 0$  and  $N \rightarrow \infty$ . This is the rigorous order of the limits. For unstable systems, mean field theory consists in interchanging both limits, taking first  $N \rightarrow \infty$ . In this case the functional integral (16) is saturated by the maximum of the exponent in the integrand, all the fluctuations around the maximizing density being suppressed by powers of  $1/N$ . Ignoring all the irrelevant constants, we can define the entropy per particle as

$$\mathcal{S} = - \int d^3r \rho(\mathbf{r}) [\log \rho(\mathbf{r}) - 1] + \frac{3}{2} \log \left( E - \frac{1}{2} L^T I^{-1} L - \Phi \right) , \quad (18)$$

with  $I$  and  $\Phi$  given by (17). The physical distribution  $\rho$  is that which maximizes  $\mathcal{S}$  under the constraint  $\int \rho(\mathbf{r}) = 1$ .

The distribution which maximizes (18) is one for which a principal inertial axis goes along the angular momentum direction, for, if not, one can define a new distribution by rotating the given distribution until the principal inertial axis with larger inertial moment coincides with the angular momentum direction. The potential energy  $\Phi$  and the pure entropical term  $\int \rho(\log \rho - 1)$  do not change, but the rotational energy has been thus lowered, and hence  $\mathcal{S}$  increases. Taking the angular momentum along the  $z$  axis, we can write

$$\mathcal{S} = - \int d^3r \rho(\mathbf{r}) [\log \rho(\mathbf{r}) - 1] + \frac{3}{2} \log \left( E - \frac{L^2}{2I_{33}} - \Phi \right). \quad (19)$$

The maximum can be obtained by taking the functional derivative respect to  $\rho$ , which gives the following integral equation:

$$\rho(\mathbf{r}) = \exp \left\{ \frac{2}{3} \beta \left[ \xi (x^2 + y^2) - \int d^3r' \phi(\mathbf{r}, \mathbf{r}') \rho(\mathbf{r}') \right] + \mu \right\}, \quad (20)$$

where  $\mu$  is the Lagrange multiplier for the constraint  $\int \rho = 1$ , and we used the notation:

$$\beta = \left( E - \frac{L^2}{2I_{33}} - \Phi \right)^{-1} \quad \text{and} \quad \xi = \frac{ML^2}{2I_{33}^2}. \quad (21)$$

#### 4 The Thirring model with angular momentum.

Long ago Thirring proposed a very simple model for a star [9]. In spite of its extreme simplicity, it mimics the main features of gravitating systems with surprisingly good success. Thirring considered the model without angular momentum. We shall see in this section that, as expected, angular momentum drastically changes the behaviour of the system.

#### 4.1 THE MODEL.

The model can be described as follows: a set of  $N$  particles are confined in a spherical volume  $V$ . Inside this volume there is a spherical interaction region (core)  $V_0$ , concentric to  $V$ . Particles outside the core ("atmosphere") do not interact, and two particles inside the core have a constant attractive potential energy. Using the step function

$$\Theta_{V_0}(\mathbf{r}) = \begin{cases} 1 & \text{if } \mathbf{r} \in V_0 \\ 0 & \text{if } \mathbf{r} \notin V_0 \end{cases} \quad (22)$$

the interaction energy is given by

$$\phi(\mathbf{r}, \mathbf{r}') = -\frac{Gm^2}{2} \Theta_{V_0}(\mathbf{r}) \Theta_{V_0}(\mathbf{r}'), \quad (23)$$

where  $m$  is the mass of a particle and  $G$  the "gravitational" constant. (We have chosen the coupling constant in analogy with a gravitating system). We can carry out the  $N \rightarrow \infty$  limit of the previous section, with  $Nm = M$  fixed. The potential energy then reads,

$$\Phi(\mathbf{r}_1, \dots, \mathbf{r}_N) = -\frac{GM^2}{2} \alpha^2, \quad (24)$$

where  $\alpha = \int_{V_0} d^3r \rho(\mathbf{r})$  is the fraction of particles inside  $V_0$ .

#### 4.2 THE MEAN FIELD EQUATIONS.

To simplify the computations, we shall consider the model in two dimensions<sup>1</sup>. Notice that in this case the factor  $3/2$  which appears in front of  $\beta$  in eq. (20) must be substituted by 1. Let us introduce the following notation:  $R$  and  $R_0$  are respectively the radius of the total volume,  $V$ , and the core,  $V_0$ , and  $\kappa = V_0/V$ . Using the dimensionless variables  $\epsilon = E/(GM^2)$  and  $\Omega = L^2/(2GM^3R^2)$ , the mean field equation (20) becomes:

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<sup>1</sup>It should not be very difficult to solve it three dimensions, but it is slightly more cumbersome and no qualitative difference is expected.

$$\rho(\mathbf{r}) = \begin{cases} \exp \left\{ \mu + \beta\alpha + \beta\xi r^2/R^2 \right\} & \text{if } r < R_0 \\ \exp \left\{ \mu + \beta\xi r^2/R^2 \right\} & \text{if } r > R_0 \end{cases} \quad (25)$$

with

$$\begin{aligned} \beta &= \left( \epsilon - \frac{\Omega}{\frac{1}{R^2} \int d^2r r^2 \rho(\mathbf{r})} + \frac{\alpha^2}{2} \right)^{-1} \\ \xi &= \frac{\Omega}{\left[ \frac{1}{R^2} \int d^2r r^2 \rho(\mathbf{r}) \right]^2}. \end{aligned} \quad (26)$$

The two self-consistency equations

$$\alpha = \int_{V_0} d^2r \rho(\mathbf{r}) \quad \text{and} \quad 1 - \alpha = \int_{V \setminus V_0} d^2r \rho(\mathbf{r}) \quad (27)$$

determine  $\alpha$  and  $\mu$ . From them, it is straightforward to derive the following equation for  $\alpha$ :

$$\log \alpha - \log(1 - \alpha) - \beta\alpha = -\beta\xi(1 - \kappa) + \log \left( \frac{1 - e^{-\beta\xi\kappa}}{1 - e^{-\beta\xi(1 - \kappa)}} \right). \quad (28)$$

Notice that  $\beta$  is positive, by definition. It is possible to eliminate  $\mu$  from (25), and then the particle distribution reads

$$\rho(\mathbf{r}) = \begin{cases} \frac{\alpha F_0}{V} \exp \left( -\beta\xi \frac{r^2}{R^2} \right) & \text{if } \mathbf{r} \in V_0 \\ \frac{(1-\alpha) F_1}{V} \exp \left( -\beta\xi \frac{r^2}{R^2} \right) & \text{if } \mathbf{r} \notin V_0 \end{cases} \quad (29)$$

where

$$F_0 = \frac{\beta\xi}{e^{\beta\xi\kappa} - 1} \quad \text{and} \quad F_1 = \frac{\beta\xi}{e^{\beta\xi} - e^{\beta\xi\kappa}}. \quad (30)$$

Eq. (28) provides the complete solution of the system. Once solved for  $\alpha$ , for different values of the energy  $\epsilon$  and the angular momentum  $\Omega$ , we can compute

the entropy  $\mathcal{S}$  and the temperature  $T$ , which, due to the fact that the mass distribution maximizes  $\mathcal{S}$ , is:

$$\frac{1}{T} = \frac{\partial \mathcal{S}[\rho, \epsilon]}{\partial \epsilon} = \beta. \quad (31)$$

Notice that  $\alpha$  is an order parameter for the collapsing phase transition, since it is the fraction of particles inside the core  $V_0$ . Notice also that (28) can have more than one solution for some values of  $\epsilon$  and  $\Omega$ . When this is the case, the true solution is that which maximizes the entropy, the others constituting metastable states.

#### 4.3 LIMITING CASES.

Let us study the solutions of (28) in some limiting cases.

- (a) If  $\Omega \rightarrow 0$ , then  $\xi \rightarrow 0$  and we recover the original Thirring model. There is a phase transition separating a low energy collapsing phase, where  $\alpha \sim 1$ , from a high energy gas phase, with  $\alpha \sim \kappa$ .
- (b) If  $\Omega \rightarrow \infty$ , then  $\xi \rightarrow \infty$ . In this case  $\alpha \rightarrow 0$  for  $\epsilon \sim 4\Omega$  and  $\alpha \rightarrow \kappa$  for  $\epsilon \gg 4\Omega$ . No collapse happens.
- (c) For fixed values of  $\Omega$  and  $\epsilon \rightarrow \infty$ , we have  $\beta \rightarrow 0$  and then  $\alpha \rightarrow \kappa$ . In the high energy region, the system behaves as a gas, irrespective of the angular momentum, and the mass distribution is homogeneous.
- (d) The most interesting case is the behaviour of the ground state for fixed values of  $\Omega$ . In the ground state the energy of the system is saturated by the rotational energy (which cannot be zero due to the angular momentum) and the potential energy, so that no energy remains for thermal motion. This is the only microscopical contribution to the thermodynamical state when  $\beta \rightarrow \infty$ , and corresponds to zero temperature. It is easy to see that the entropy goes to  $-\infty$  like  $-\log \beta$ , hence the non-thermal (rotational + potential) energy must be at its minimum, for, if not, the system can change its mass distribution in such a way that some amount of thermal energy  $1/\beta$  appears, rising the entropy. In the next subsection we shall see what is the structure of the ground state for the different values of  $\Omega$ .

#### 4.4 THE GROUND STATE.

As we just discussed, when  $\beta \rightarrow \infty$  the non-thermal energy

$$\epsilon_{nt} = \frac{\Omega}{\frac{1}{R} \int d^2r r^2 \rho(\mathbf{r})} - \frac{\alpha^2}{2} \quad (32)$$

must be at its minimum. It is possible to minimize the rotational energy without modifying the potential energy, just taking a mass distribution with fixed  $\alpha$  which maximizes the inertial moment. Obviously, this mass distribution correspond to a fraction  $\alpha$  of the particles in the outer layer of the core, and the remaining  $1 - \alpha$  fraction in the outer layer of the system. The dimensionless inertial moment is then  $(1/R^2) \int d^2r r^2 \rho(\mathbf{r}) = 1 - \alpha(1 - \kappa)$ . Hence, the non-thermal energy is

$$\epsilon_{nt} = \frac{\Omega}{1 - \alpha(1 - \kappa)} - \frac{\alpha^2}{2}, \quad (33)$$

The minimum of the above function is at either  $\alpha = 0$ ,  $\alpha = 1$ , or at one solution of

$$\frac{\Omega(1 - \kappa)}{[1 - \alpha(1 - \kappa)]^2} - \alpha = 0. \quad (34)$$

Eq. (34) is equivalent to a real polynomial equation of third degree, and therefore has either one or three real solutions. Since the first term in (34) has a pole at  $\alpha = 1/(1 - \kappa) > 1$ , there is always a solution at  $\alpha > 1/(1 - \kappa)$ , which is non-physical since  $\alpha$  must belong to  $[0, 1]$ . Besides this non-physical solution, when  $\Omega$  is small (34) has two more solutions. For  $\Omega \ll 1$

$$\alpha_{max} \approx \Omega(1 - \kappa) \quad \text{and} \quad \alpha_{min} \approx \frac{1}{1 - \kappa} \left( 1 - \sqrt{\Omega(1 - \kappa)} \right). \quad (35)$$

From the second derivative of (34) we see that there is an inflection point at

$$\alpha_{ip} = \frac{1 - \sqrt[3]{\Omega(1 - \kappa)^2}}{1 - \kappa}, \quad (36)$$

and  $\alpha_{max}$  correspond to a local maximum and  $\alpha_{min}$  to a local minimum. We always have  $\alpha_{max} \leq \alpha_{ip} \leq \alpha_{min}$ . Taking the derivative of (34) respect to  $\Omega$  - with  $\alpha = \alpha_{min}$  - we easily see that  $\alpha_{min}$  is a monotonically decreasing function of  $\Omega$ . Analogously, introducing  $\alpha = \alpha_{min}$  in (33), taking the derivative respect to  $\Omega$  and using (34), we see that the value of  $\epsilon_{nt}$  at its local minimum  $\alpha_{min}$  is a monotonically increasing function of  $\Omega$ .

Let us study the absolute minimum of (33) for  $0 \leq \alpha \leq 1$ . The three candidates are  $\alpha = 0, 1$  or  $\alpha_{min}$ . For small  $\Omega$ , from (35) we learned that  $\alpha_{min} > 1$ , and, since  $\epsilon_{nt}(0) = \Omega$  and  $\epsilon_{nt}(1) = \Omega/\kappa - 1/2$ , the absolute minimum correspond to  $\alpha = 1$ . But when  $\Omega$  grows  $\epsilon_{nt}(1)$  increases faster than  $\epsilon_{nt}(0)$ , and  $\alpha_{min}$  will reach the physical region, since it decreases. The following three possibilities define three critical values of  $\Omega$ :

- (i) The relative minimum  $\alpha_{min}$  reaches the physical region. The critical value  $\Omega_1$  is given by  $\alpha_{min}(\Omega_1) = 1$ . From (34) we obtain  $\Omega_1 = \kappa^2/(1 - \kappa)$ .
- (ii) The relative minimum  $\alpha_{min}$  ceases to be the absolute minimum. This happens for  $\Omega = \Omega_2$ , when  $\epsilon_{nt}(\alpha_{min}) = \epsilon_{nt}(0)$ . We have the two equations

$$\alpha_{min} = \frac{\Omega_2(1 - \kappa)}{[1 - \alpha_{min}(1 - \kappa)]^2} \quad \text{and} \quad \Omega_2 = \frac{\Omega_2}{1 - \alpha_{min}} - \frac{\alpha_{min}^2}{2}.$$

The solution is  $\Omega_2 = 1/[8(1 - \kappa)^2]$  and  $\alpha_{min} = 1/[2(1 - \kappa)]$ .

- (iii) The value of  $\epsilon_{nt}$  at  $\alpha = 0$  surpasses that at  $\alpha = 1$ . This happens when  $\Omega = \Omega_3$ , having  $\epsilon_{nt}(0) = \epsilon_{nt}(1)$ . Clearly  $\Omega_3 = \kappa/[2(1 - \kappa)]$ .

The ground state behaviour depends on the ordering of these critical  $\Omega$ , which in its turn depends on  $\kappa$ . For  $\kappa < 1/2$  we have  $\Omega_1 < \Omega_2, \Omega_3$ . Then the ground state is characterized by complete collapse if  $\Omega < \Omega_1$ , with all the particle inside the core. For  $\Omega_1 < \Omega < \Omega_2$  there is an incomplete, with most - but not all - the particle in the core. The value of  $\alpha$  varies continuously from  $\alpha = 1$  at  $\Omega_1$  to  $\alpha = 1/[2(1 - \kappa)]$  at  $\Omega_2$ . For  $\Omega > \Omega_2$  the ground state has an empty core ( $\alpha = 0$ ). Notice the jump of  $\alpha$  at  $\Omega_2$ .

For  $\kappa > 1/2$  we have  $\Omega_3 < \Omega_1$  and  $\Omega_3 < \Omega_2$ . In this case, the ground state for  $\Omega < \Omega_3$  is characterized by a complete collapse, with all the particles inside the core. For  $\Omega > \Omega_3$  no collapse happens ( $\alpha = 0$ ). In fig. 1 we display the behaviour of the ground state for both  $\kappa < 1/2$  and  $\kappa > 1/2$ .

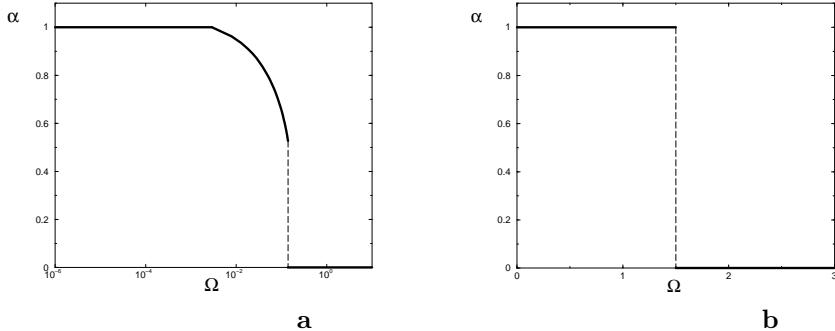


Figure 1: The structure of the ground state as a function of  $\Omega$  for a)  $\kappa < 1/2$  (observe the logarithmic scale in abscissas) and b)  $\kappa > 1/2$ .

#### 4.5 PHASE DIAGRAM.

For the discussion of the model at finite temperatures we consider the most interesting case,  $\kappa < 1/2$  (atmosphere bigger than the core). The numerical solutions displayed in the figures are for  $\kappa = 1/(e^3 - 1) \approx 0.0524$ . Exactly as for the ground state, there are three different phases, which can be distinguish by the behaviour of the order parameter  $\alpha$ , which is displayed in fig. 2 as a function of the energy, for three values of  $\Omega$ , representative of each phase. Figs. 3 and 4 display the temperature and the entropy for the same values of  $\Omega$ . It can be clearly seen the correlation between the collapsing transition and the anomalies in the thermodynamical quantities. Let us discuss in detail what is happening on each phase.

- (1)  $\Omega < \Omega_1$ . There is a complete collapse at low energies, with  $\alpha \approx 1$ . The gas phase at high energies is separated from the collapsing phase by an interval of energies with negative specific heat. Qualitatively, the model is similar to the original ( $\Omega = 0$ ) Thirring model. The entropy shows the characteristic convex intruder in the energy interval where the two phases coexist. (Notice that, since the system is not thermodynamically stable, van Hove's theorem [13] does not apply).
- (2)  $\Omega_1 < \Omega < \Omega_2$ . At low energies the collapse is not complete, with  $\alpha < 1$ . For values of  $\Omega$  near  $\Omega_1$  the thermodynamical quantities are qualitatively equal to those in the low angular momentum phase (negative specific heat). When  $\Omega$  is larger, eq. (28) has more than one solution. For energies larger than a certain value, one of the new solutions becomes the absolute maximum of the entropy. This is the origin of the jumps in  $\alpha$  and in  $T$  that

can be seen in figs. 2 and 3, for  $\Omega = 0.05$ . Notice that the jump occurs after a region with negative specific heat. For  $\Omega$  still larger, the jump appears before the specific heat becomes negative. However, although in the last cases the specific heat is positive everywhere, the entropy still has a convex intruder, due to the kink originated by the jump in the temperature.

(3)  $\Omega > \Omega_2$ . There is no collapse at low energies. The specific heat is positive, smooth and increases monotonically with the energy. The entropy has no convex intruder.

From the above discussion, it becomes manifest the correlation between the collapsing phase transition and the anomalies in the caloric curve ( $T$  versus  $\epsilon$ ) and in the entropy. The phase diagram in the plane  $(\epsilon, \Omega)$  is displayed in fig. 5. Obviously, there is a forbidden region, where the system cannot be since a minimal rotational energy is required to keep angular momentum constant. The boundary between the forbidden and allowed regions is the  $T = 0$  isotherm. Below the forbidden region, there are two phases<sup>2</sup> separated by a transition region, which is defined through the Maxwell construction. Since both phases are qualitatively different, they must be analytically separated. This means that the critical point must be at zero temperature (i.e., on the curve of minimal energy). And, in its turn, this implies that the critical point must be at  $\Omega_c = \Omega_2$ , since this is the point on the zero temperature line where the collapse disappear. This is what actually happens, as can be seen in fig. 5.

For fixed  $\Omega$ , the difference of the energies limiting the transition region defines the latent heat. We see that the phase transition is first order everywhere, with non-vanishing latent heat. From the analysis of eq. (28) when  $\beta \rightarrow \infty$  and  $\Omega \rightarrow \Omega_2$  one concludes<sup>3</sup> that the latent heat vanishes as  $\Omega_2 - \Omega$ . Therefore, at the critical point,  $\Omega_2$ , the latent heat disappears. Since the specific heat is continuous at this point, we cannot say that the phase transition becomes second order; indeed, the order parameter  $\alpha$  jumps at  $\Omega_2$ .

#### 4.6 DISCUSSION.

Now, we want to address the question of whether the phase diagram described in the previous subsection shares its main properties with those of more

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<sup>2</sup>The two collapsing phases,  $\Omega < \Omega_1$  and  $\Omega_1 < \Omega < \Omega_2$ , are distinguished only by the behaviour of the order parameter at low energies. The thermodynamical quantities, however, behave in a similar way in both cases. Therefore, in the phase diagram we shall not distinguish the two low energy phases.

<sup>3</sup>After a careful study of the possible solutions, their entropies, etc., which is not reported here in detail.

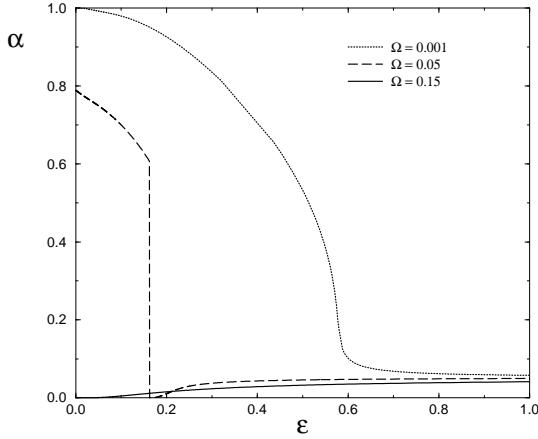


Figure 2: The collapsing order parameter  $\alpha$  versus the energy for  $\Omega$  values on each of the three phases. To facilitate comparisons, we have redefined the energy by a shift, such that the minimum energy is  $\epsilon = 0$  for all  $\Omega$ .

realistic models, or whether these are merely a consequence of the extreme simplicity of the model. We can give at least a partial answer. The model considered here is unrealistic mainly due to the fact that the particles can only collapse in a fixed region. Therefore, there is no place here for more complicated regimes like multifragmentation.

The collapsing phase, at low angular momentum  $L$ , with all - or most - the particles forming a cluster, is expected to be present in realistic problems. Of course, the size of the cluster will increase with angular momentum. For large energies both the potential and rotational energy are negligible and we shall have a gas phase. There will be then a transition region separating this two phases, like in fig. 5. The difference may be, however, that in realistic cases these two phases might be connected through intermediate regimes like multifragmentation. It is clear that for large enough angular momentum the system will prefer to collapse into several bodies rather than in a single one. The critical point, if any, might be located at non-zero temperature. If this is the case, one could expect a true second order phase transition, with divergent specific heat and large fluctuations of some local order parameter, like the density. Needless to say, these are questions which deserve a careful investigation.

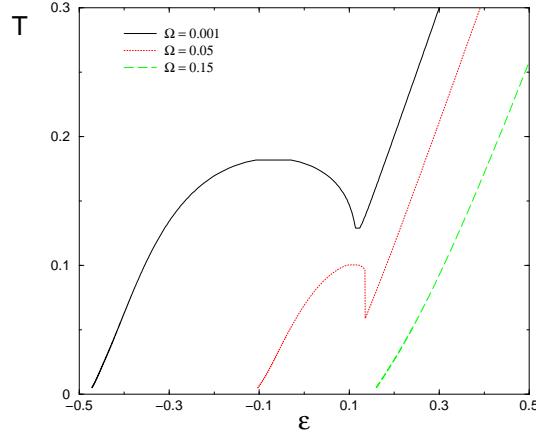


Figure 3: The microcanonical temperature versus the energy for three values of  $\Omega$ , one on each phase of the system.

## 5 Conclusions.

Let us briefly summarize the results presented in this work. First, we found a non-singular expression for the microcanonical distribution of systems with conserved, non-vanishing angular momentum. It was obtained by integrating out the momenta in the phase space, and it is suitable for Monte Carlo simulations, as well as for mean field computations. We presented a derivation of the mean field equations for systems of classical particles interacting through a two body unstable potential, taking into account the conservation of angular momentum.

We discussed, as an application, the properties of the phase diagram of a simple model of gravitating particles with non-vanishing angular momentum. We found the exact solution in the mean field approach, which served to illustrate how the phase diagram of a physical system can be modified by the conservation of the angular momentum. In the case under interest, the angular momentum  $L \sim \sqrt{\Omega}$  determines the existence of three phases, separated by two critical values  $\Omega_1$  and  $\Omega_2$ . At low energies all particles condense in a small region if  $\Omega < \Omega_1$ . For  $\Omega_1 < \Omega < \Omega_2$  the collapse at low energy is incomplete with most - but not all - particles condensed in a small region. At high energies, the system behaves like a gas irrespective of the value of  $\Omega$ . The collapsing-gas phase transition for  $\Omega < \Omega_2$  is accompanied by an anomaly in the caloric curve ( $T$  versus  $\epsilon$ ), which reflects the fact that the entropy is not concave in an energy

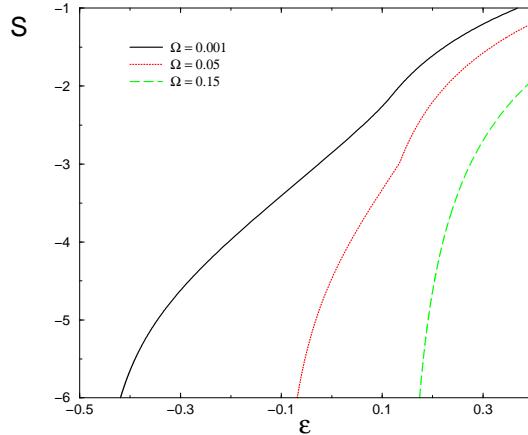


Figure 4: The entropy versus the energy for three values of  $\Omega$ , one on each phase of the system. The convex intruder for the two smaller  $\Omega$  can be appreciated. For  $\Omega = 0.15$  the entropy is concave.

interval. This interval coincides with that where the collapsing-gas transition takes place. Through the Maxwell construction, it is possible to define precisely this transition region and a latent heat, which shows that the transition can be classified as first order. The latent heat disappear continuously at  $\Omega = \Omega_2$  and  $T = 0$ . This reflects the fact that both phases, being qualitatively different, cannot be analytically connected in the phase diagram. At  $\Omega > \Omega_2$  there is neither collapse nor anomaly in the thermodynamical quantities. We argued that, with more realistic potentials, an intermediate regime between the collapse and gas phases - multifragmentation perhaps - should exist. Therefore, it might be possible that the system pass smoothly from one phase to the other. Therefore, the critical point might be located at non-zero temperature, showing the typical behaviour of a second order phase transition.

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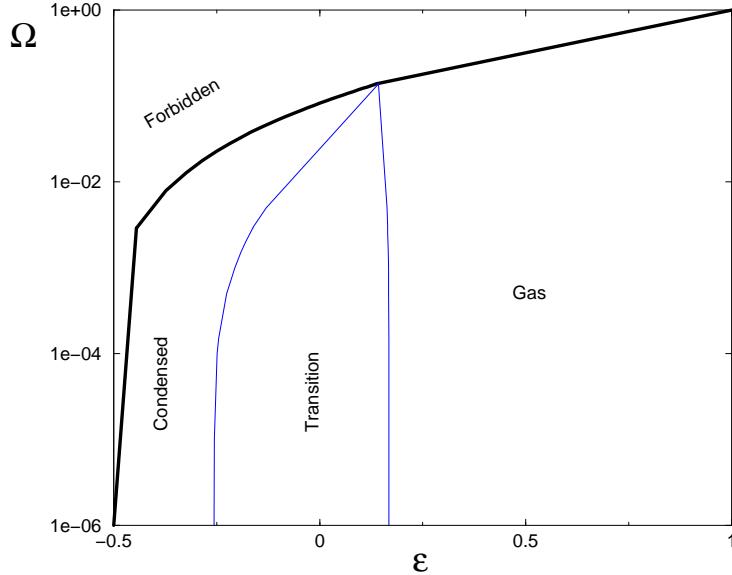


Figure 5: The phase diagram of the model. Notice the logarithmic scale in the ordinate axis. The thick line corresponds to the  $T = 0$  isotherm.

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